# A Dynamic Model of Two-layered Forest

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# Basic Framework

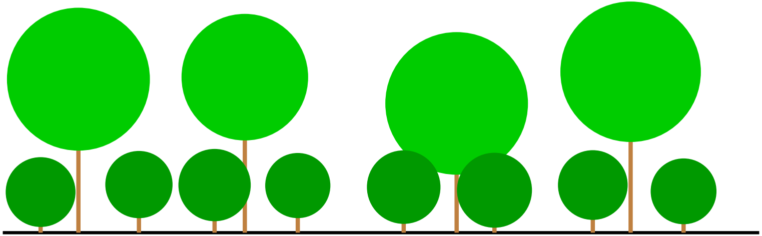


Figure: Forest of two layers

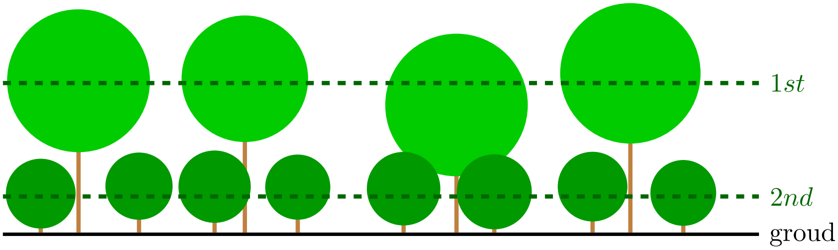



Figure: Forest of two layers. Dashed lines show the layers

## Definition

$T \in (0, 1]$: Density(Cover) of the 1st layer
$B \in (0, 1]$: Density(Cover) of the 2nd layer

## Basic framework

$$\frac{\mathrm{d}T}{\mathrm{d}t} = f(T, B, \text{parameters})$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = g(T, B, \text{parameters})$$

## Lotka-Volterra model

$$\frac{\mathrm{d}T}{\mathrm{d}t} = r_T \cdot T - \beta_{BT} \cdot B \cdot T$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \underbrace{r_B \cdot B}_{Growth} - \underbrace{\beta_{TB} \cdot T \cdot B}_{Death}$$

Considering the following process:

- Growth of $T$ is influenced by $B$, the light conditions in the $B$-layer ($I_B$);
- Death of $T$ related to $T$ and light condition in $T$-layer ($I_T$).
- Growth of $B$ is influenced by $S(T, B)$, the light conditions in the $S$-layer ($I_S$);
- Death of $B$ related to $B$ and light condition in $B$-layer ($I_B$).

**Model framework**

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= \text{Growth}(T, B, I_B) - \text{Death}(T, I_T) \\ \frac{\mathrm{d}B}{\mathrm{d}t} &= \text{Growth}(B, S(T, B), I_S) - \text{Growth}(T, B, I_B) - \text{Death}(B, I_B) \end{aligned} \tag{1}$$

# Dynamic Model of Two-layered Forest

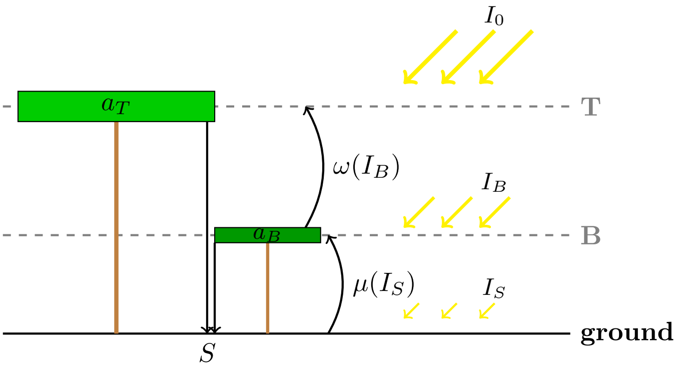



Figure: Diagram of parameters and process

**Definition**

$I_0$: Total light intensity in unit area ($I_0 \in (0,1]$);
$I_T, I_B, I_S$: Light intensity in layer $T$, layer $B$ and ground layer.

$$I_T = I_0 \tag{2}$$

$$I_B = I_T \cdot (1 - a_1 T) \tag{3}$$

$$I_S = I_B \cdot (1 - a_2 B) \tag{4}$$

where $a_1$, $a_2$ are light absorption coefficient, given $I_0$ (Here we set $a_1 = 0.9$, $a_2 = 0.5$).

# Transition between layers

## Model framework

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \text{Growth}(T, B, I_B) - \text{Death}(T, I_T)$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = \text{Growth}(B, S(T, B), I_S) - \text{Growth}(T, B, I_B) - \text{Death}(B, I_B)$$

The function $\text{Growth}_T(B, T, I_B)$ represent the transition speed from $B$-layer to $T$-layer, influenced by light intensity $I_B$. Similarly, $\text{Growth}_B(S(T, B), B, I_S)$ represents the transition speed from seeds to the $B$-layer, influenced by light intensity $I_S$.

### Growth-T

$$\text{Growth}_T(B, T, I_B) = B \cdot \omega(I_B) \cdot (1 - T)$$

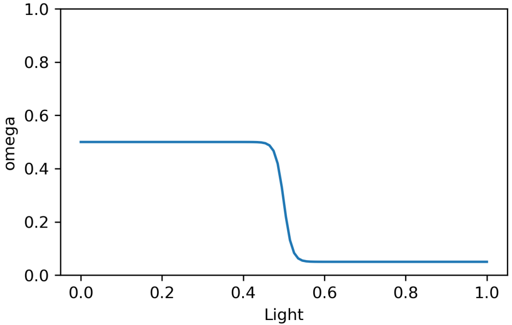
where

$\omega(I_B)$ is the transition rate from $B$ to $T$ with light intensity $I_B$;

### B-T transition kernel

$$\omega(I_B) = \frac{\omega_0 - \omega_1}{1 + e^{100 \cdot (I_B - c_B)}} + \omega_1 \qquad (5)$$

where $\omega_0$ and $\omega_1$ are the maxmum and minimum limit of the rate (here set $\omega_0 = 0.5, \omega_1 = 0.05$), $c_B = 0.5$ is the center of change.



### Growth-B

$$\text{Growth}_B(S(T,B), B, I_S) = S(T,B) \cdot \mu(I_S) \cdot (1 - B)$$

where
$\mu(I_S)$ is the transition rate from seeds to $B$ with light intensity $I_S$.

### Definition

$S(T,B)$: the density of seeds.
Here we set:

$$S = I_T \cdot T + I_B \cdot B \tag{6}$$

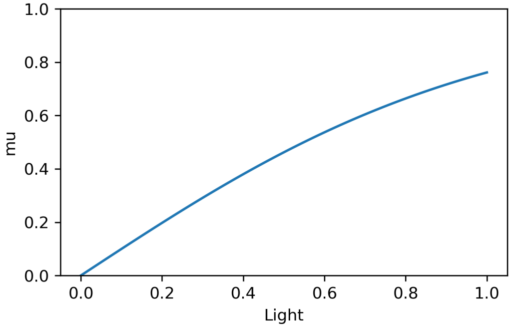
.

### S-B transition kernel

$$\mu(I_S) = \frac{e^{2 \cdot I_S} - 1}{e^{2 \cdot I_S} + 1} \tag{7}$$

.

The transition from seed to $B$-layer increases with light intensity of the ground.



# Light competition in same layer

## Model framework

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \text{Growth}(T, B, I_B) - \text{Death}(T, I_T)$$
$$\frac{\mathrm{d}B}{\mathrm{d}t} = \text{Growth}(B, S(T, B), I_S) - \text{Growth}(T, B, I_B) - \text{Death}(B, I_B)$$

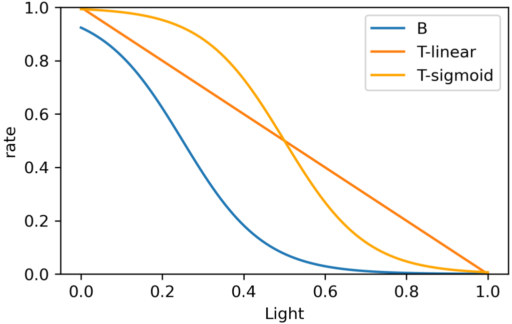
For the decline of layers, $\text{Death}_T = T \cdot D_T(I_T)$, $\text{Death}_B = B \cdot D_B(I_B)$, where $D_T$ and $D_B$ represent the effect of light.

For function $D_T$, linear function is enough, also, sigmoid kernel could be used.

$$D_T = 1 - I_T \tag{8}$$

(or, $D_T = \frac{1}{1+e^{10(I_T-0.75)}}$)
But for $D_B$, to ensure that $B$ has higher tolerance to light limitation and the death rate remains $[0, 1]$, sigmoid function is necessary.

$$D_B = \frac{1}{1 + e^{10(I_B-0.25)}} \tag{9}$$

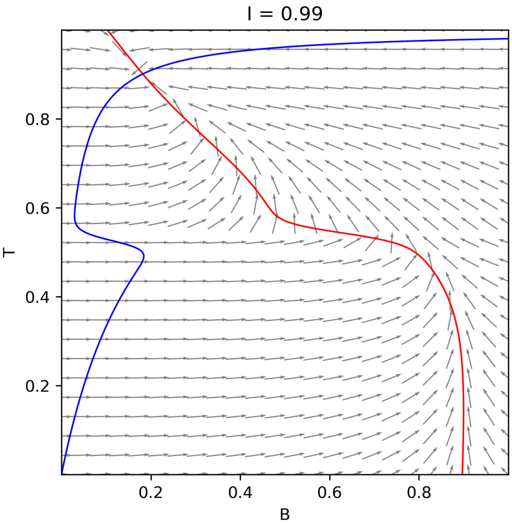


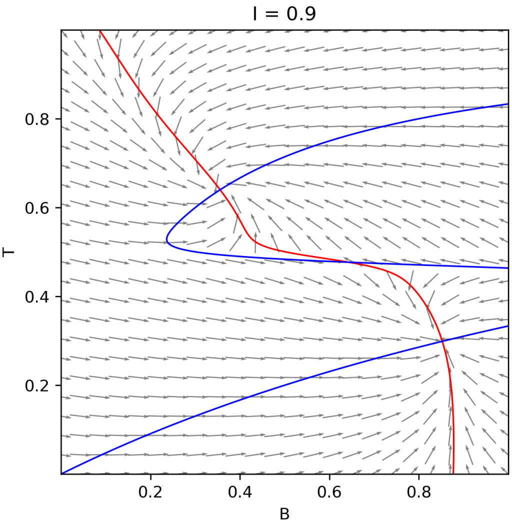
# Full model

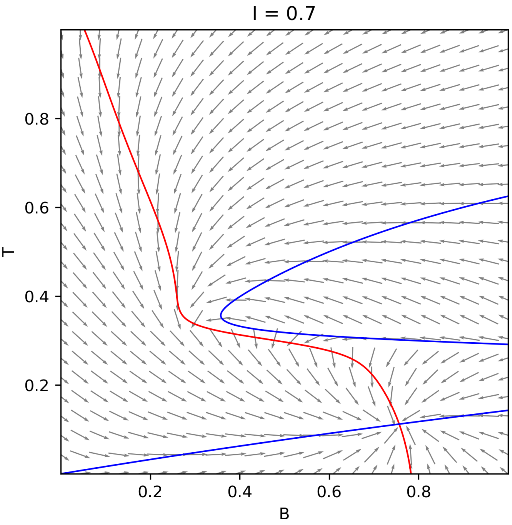
## Full Model

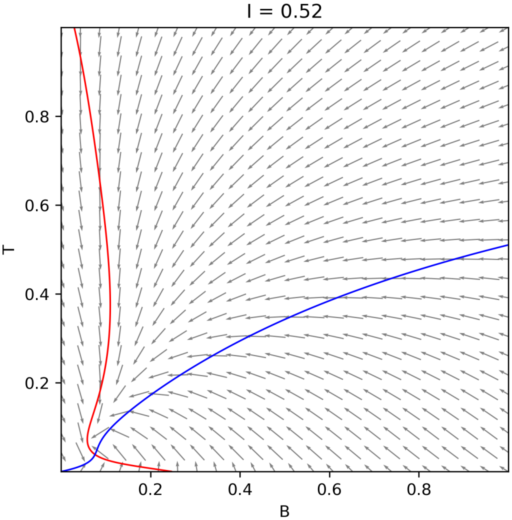
$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &= B \cdot \omega(I_B) \cdot (1 - T) - T \cdot D_T(I_T) \\
\frac{\mathrm{d}B}{\mathrm{d}t} &= S_{(T,B)} \cdot \mu(I_S) \cdot (1 - B) - B \cdot \omega(I_B) \cdot (1 - T) - B \cdot D_B(I_B)
\end{aligned} \tag{10}
$$

# Simulation and Analysis


I = 0.99
T
B
0.2
0.4
0.6
0.8
0.2
0.4
0.6
0.8

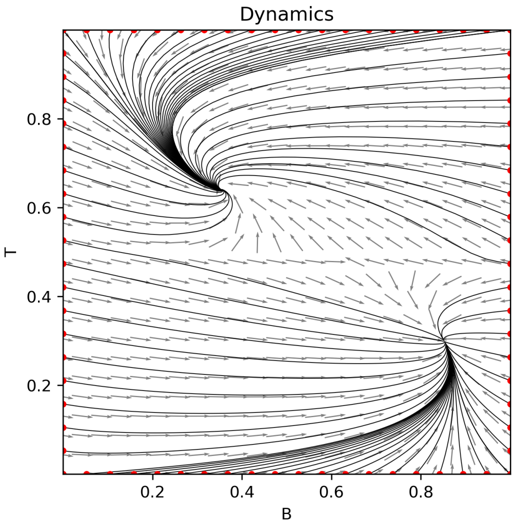

I = 0.9
0.8
0.6
0.4
0.2
T
0.2
0.4
0.6
0.8
B


I = 0.7
0.8
0.6
0.4
0.2
T
0.2
0.4
0.6
0.8
B


I = 0.52
0.8
0.6
0.4
0.2
T
0.2
0.4
0.6
0.8
B

## Edge condition

$$
\begin{aligned}
\left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{T=0} &= B \cdot \omega(I_0) > 0 \\
\left.\frac{\mathrm{d}T}{\mathrm{d}t}\right|_{T=1} &= -D_T(I_0) < 0 \\
\left.\frac{\mathrm{d}B}{\mathrm{d}t}\right|_{B=0} &= S_{(T,0)} \cdot \mu(I_S) > 0 \\
\left.\frac{\mathrm{d}B}{\mathrm{d}t}\right|_{B=1} &= -(\omega(I_B) \cdot (1-T) + D_B(I_B)) < 0
\end{aligned}
$$


Dynamics
0.8
0.6
0.4
0.2
T
0.2
0.4
0.6
0.8
B

## Equilibrium state

$$Equation(10), \frac{\mathrm{d}T}{\mathrm{d}t} = 0, \frac{\mathrm{d}B}{\mathrm{d}t} = 0 \Rightarrow$$

$$0 = B^* \cdot \omega(I_B) \cdot (1 - T^*) - T^* \cdot D_T(I_T) \tag{11}$$

$$0 = S_{(T,B)} \cdot \mu(I_S) \cdot (1 - B^*) - B^* \cdot \omega(I_B) \cdot (1 - T^*) - B^* \cdot D_B(I_B) \tag{12}$$

$$\Rightarrow B^* = \frac{T^*(1 - I_0)}{\omega(I_B)(1 - T^*)} \tag{13}$$

$$\Rightarrow B^*, T^*$$

# Stability with numerical solution

For given $I_0 \in [0, 1]$, calculate $B^*, T^*$. Then compute the Jacob matrix of the equilibrium state.

$$J = \begin{vmatrix} \frac{\partial}{\partial T}\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right) & \frac{\partial}{\partial B}\left(\frac{\mathrm{d}T}{\mathrm{d}t}\right) \\ \frac{\partial}{\partial T}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right) & \frac{\partial}{\partial B}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right) \end{vmatrix}$$

Calculate the eigen value $(\lambda_1, \lambda_2)$ of matrix $J$. $(B^*, T^*)$ is stable only if $\Re(\lambda_i) < 0$

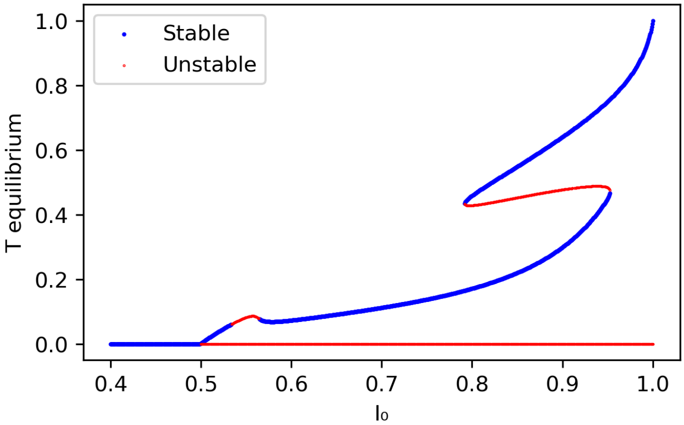



Figure: System stability change along $I_0$, here shows the cover of top layer. The system exhibits bistability over the interval $\sim (0.8, 0.95)$.

# Approximation of stability

Notice that equation (13) depends on the function (5), which can be approximated by a piecewise function:

$$(5) \Rightarrow \omega \approx \begin{cases} \omega_0, & I_B < 0.5 \\ \omega_1, & I_B > 0.5 \end{cases}$$

same to which, function(9) can be approximated by:

$$(9) \Rightarrow D_B \approx \begin{cases} 1 - 2I_B, & I_B < 0.5 \\ 0, & I_B > 0.5 \end{cases}$$

$I_B < 0.5$

$$\begin{cases} B^* = \frac{T^*(1-I_0)}{0.5(1-T^*)} \\ S_{(T,B)}\mu(I_S)(1-B^*) = T^*(1-I_0) + B^*(1-2I_B) \end{cases} \Rightarrow \hat{T_-^*}(I_0)$$

$I_B > 0.5$

$$\begin{cases} B^* = \frac{T^*(1-I_0)}{0.05(1-T^*)} \\ S_{(T,B)}\mu(I_S)(1-B^*) = T^*(1-I_0) \end{cases} \Rightarrow \hat{T_+^*}(I_0)$$

$$I(1 - a_1\hat{T^*_-}(I)) < 0.5 \Rightarrow I_-$$
$$I(1 - a_1\hat{T^*_+}(I)) < 0.5 \Rightarrow I_+$$

Then bistability maintains approximately in region $(I_-, I_+)$

# Further work

# Further work

- Further review the literature and incorporate real-world data to determine the parameter range of the model.
- Test the model's sensitivity to parameters, identify key parameters, and determine the parameter space for bistability.
- Approximate and simplify the model, attempt to derive an analytical solution, and conduct further analytical analysis.
- Constrain the parameter space based on life-history trade-offs and analyze the system's dynamical characteristics.
- Introduce a diffusion term and simulate the system in a spatial domain with a light gradient.

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